A Minimum of Alignment Energy

In Section 5 we claimed that the normalized solution $\psi = \tilde{\psi}/\|\tilde{\psi}\|$ of

$$(A - \lambda(t)I)\tilde{\psi} = \phi$$

for $\lambda(t) \in (-\infty, \lambda_1)$ yields a global minimum of $E_{s,t}(\psi)$ over all $\|\psi\| = 1$. Here we prove this fact and show the relationship between λ and t. For simplicity we assume throughout that $\|\phi\| = 1$. Furthermore, we will only use the real part of the product $\langle\langle .,. \rangle\rangle$ in this section.

At a constrained minimum the constraint must be fulfilled and the gradients of energy and constraint must be parallel, *i.e.*, there exists a Lagrange multiplier $\lambda/2$ such that

$$A\psi - \tfrac{t}{1-t}\phi = \lambda(t)\psi. \tag{20}$$

Substituting $\psi = \tilde{\psi}/\|\tilde{\psi}\|$ we find it to solve this equation for $t = (1 + \|\tilde{\psi}\|)^{-1}$.

Notice now how for $t \to 0$, ψ approaches the eigenvector belonging to the smallest eigenvalue λ_1 (we are working with a positive semidefinite Hermitian operator), since we are looking for a minimum of the energy. To see the behavior for $t \to 1$ consider the inner product of Eq. (20) with ψ

$$\langle\langle A\psi, \psi \rangle\rangle = \lambda(t) + \tfrac{t}{1-t}\langle\langle \phi, \psi \rangle\rangle.$$

Together with

$$\langle\langle A\psi, \psi \rangle\rangle \leq \langle\langle A\phi, \phi \rangle\rangle,$$

which we will show in a moment, we get

$$\lambda(t) \leq \langle\langle A\phi, \phi \rangle\rangle - \tfrac{t}{1-t}\langle\langle \phi, \psi \rangle\rangle.$$

As $t \to 1$ the scalar product $\langle\langle \phi, \psi \rangle\rangle \to 1$ and so $\lambda(t) \to -\infty$.

To see that

$$\langle\langle A\psi, \psi \rangle\rangle \leq \langle\langle A\phi, \phi \rangle\rangle$$

we consider that for a minimizer ψ

$$\langle\langle A\psi, \psi \rangle\rangle - \tfrac{t}{1-t}\langle\langle \phi, \psi \rangle\rangle \leq \langle\langle A\phi, \phi \rangle\rangle - \tfrac{t}{1-t}\|\phi\|^2.$$

With $\|\phi\| = 1$ we get

$$\langle\langle A\psi, \psi \rangle\rangle \leq \langle\langle A\phi, \phi \rangle\rangle - \tfrac{t}{1-t}(1 - \langle\langle \phi, \psi \rangle\rangle)$$

and since $\langle\langle \phi, \psi \rangle\rangle \leq 1$ the desired bound.

Together we see that as $t \to 0$, $\lambda(t) \to \lambda_1$ and for $t \to 1$, $\lambda(t) \to -\infty$.

Finally we show that there exists a strictly monotone analytic function $t(\lambda) : (-\infty, \lambda_1) \to (0, 1)$.

Since the resolvent

$$R(\lambda) = (A - \lambda I)^{-1}$$

is a bounded self-adjoint operator which is an analytic function of λ so is

$$\psi(\lambda) = \frac{R(\lambda)\phi}{\|R(\lambda)\phi\|}$$

and $t(\lambda) = (1 + \|R(\lambda)\phi\|^{-1})$.

Now assume by contradiction that $t(\lambda)$ is not strictly monotone. Then there exists a $\lambda_0 \in (-\infty, \lambda_1)$ for which $t'(\lambda_0) = 0$. Taking the derivative at λ_0 of Eq. (20) we find

$$(A - \lambda_0 I)\psi'(\lambda_0) - \psi(\lambda_0) = 0.$$

Taking the inner product of this equation with $\psi'(\lambda_0)$ the second term vanishes (since $\|\psi\| = 1$) leading to $\langle\langle (A - \lambda_0 I)\psi'(\lambda_0), \psi'(\lambda_0) \rangle\rangle = 0$, which contradicts the positive definiteness of $A - \lambda_0 I$.

Together these results show that solving

$$(A - \lambda(t)I)\tilde{\psi} = \phi$$

for a choice of $\lambda \in (-\infty, \lambda_1)$ finds a global minimizer of $E_{s,t}(\psi)$ for $t(\lambda)$, a strictly monotone function with range $(0, 1)$. For $\lambda(t) \to \lambda_1$, $t \to 0$ (*i.e.*, no alignment) while for $\lambda(t) \to -\infty$, $t \to 1$ (*i.e.*, no smoothing).

B Poincaré-Hopf

In this Section we assign to every n-direction field ψ an index $\mathrm{index}_t\,\psi \in \{-1, 0, 1\}$ on each triangle t. This is how we locate the singularities and label them as positive or negative. Under some smoothness assumption on the surface we will prove a discrete version of the Poincaré-Hopf theorem. A proof in the smooth case can be found in [Ray et al. 2008].

Recall that we describe the parallel transport of n-vectors by transport coefficients r_{ij} defined for each oriented edge e_{ij}. They can be thought of as an angle valued 1-form, *i.e.*, they satisfy $r_{ji} = r_{ij}^{-1}$. For each face t_{ijk} there is a unique real number $\Omega_{ijk} \in (-\pi, \pi)$ such that

$$r_{ij} r_{jk} r_{ki} = e^{\imath\Omega_{ijk}}.$$

We call Ω the curvature 2-form of the transport.

For a face $t = t_{ijk}$ we call the total curvature pushed into the triangle from its three vertices the *geometric curvature*

$$\sigma_t := s_i \alpha_i^{jk} + s_j \alpha_j^{ki} + s_k \alpha_k^{ij} - \pi$$

of t. A triangle mesh is called n-smooth if for each face we have

$$|\sigma_t| < \tfrac{\pi}{n}.$$

On an n-smooth mesh we have $\Omega_t = n\sigma_t$ and therefore a Gauß-Bonnet type theorem:

$$\sum_{t \in T} \Omega_t = 2n\pi\chi.$$

Here χ is the Euler characteristic of the mesh.

Let now ψ be an n-direction field given by complex numbers u_i of norm one for each vertex. Then for each edge e_{ij} we define the *rotation angle* of ψ as the unique number $\omega_{ij} \in (-\pi, \pi)$ such that

$$u_j = e^{\imath\omega_{ij}} r_{ij} u_i.$$

Then for each face $t = t_{ijk}$ we have $\omega_{ij} + \omega_{jk} + \omega_{ki} + \Omega_{ijk} \in (-4\pi, 4\pi)$ and $e^{\imath(\omega_{ij}+\omega_{jk}+\omega_{ki}+\Omega_{ijk})} = 1$. This means that for every face we have an integer

$$\mathrm{index}_t\,\psi := \tfrac{1}{2\pi}(\omega_{ij} + \omega_{jk} + \omega_{ki} + \Omega_{ijk}) \in \{-1, 0, 1\}.$$

In the language of Discrete Exterior Calculus this could be expressed as

$$\mathrm{index}\,\psi = \tfrac{1}{2\pi}(d\omega + \Omega)$$

If we sum this equation over all faces, the rotation angles cancel and we are left with the

Discrete Poincaré-Hopf theorem: On an n-smooth closed triangle mesh the index sum of every n-direction field equals $2n\chi$ where χ is the Euler characteristic of the mesh.

C Difference of Anti-Holomorphic and Holomorphic Energy

Here we show that for any smooth n-vector field ψ we have $E_A(\psi) - E_H(\psi) = \frac{1}{2}\int_M nK|\psi|^2 dA - \frac{1}{2}\int_{\partial M} \mathrm{Im}\langle\nabla\psi,\psi\rangle$. First we spell out all the requirements, each of which is fulfilled in our setting, of the theorem and a few facts that we will use during the derivation.

Let M be an oriented surface with Riemannian metric. Denote by J the complex structure on M. Suppose we have a complex line bundle L over M, *i.e.*, L is a 2-dimensional real vector bundle with a complex structure J. Our ψ are sections of such a line bundle. Assume that L comes with a complex connection ∇, meaning ∇ is compatible with J: $\nabla_X(J\psi) = J\nabla_X\psi$ for all sections ψ of L and all vector fields X. Then a section ψ is called holomorphic resp. anti-holomorphic if

$$0 = \bar\partial_X\psi := \tfrac{1}{2}(\nabla_X\psi + J\nabla_{JX}\psi)$$
$$0 = \partial_X\psi := \tfrac{1}{2}(\nabla_X\psi - J\nabla_{JX}\psi).$$

If ψ is a section of L, at each point $p \in M$ the linear map $\nabla\psi$ from the 2-dimensional real vector space T_pM into the 2-dimensional real vector space L_p can be decomposed into a complex linear part $(\partial\psi)_p$ and an anti-linear part $(\bar\partial\psi)_p$ and we can write

$$\nabla = \partial + \bar\partial.$$

Now suppose in addition that each L_P comes with a Hermitian scalar product $\langle .,. \rangle_p$ that is invariant under J and that ∇ is a metric connection, *i.e.*,

$$X\langle\psi,\varphi\rangle = \langle\nabla_X\psi,\varphi\rangle + \langle\psi,\nabla_X\varphi\rangle.$$

Replacing here X by some Lie bracket $[X,Y]$ we obtain

$$\begin{aligned}
&\langle\nabla_{[X,Y]}\psi,\varphi\rangle + \langle\psi,\nabla_{[X,Y]}\varphi\rangle = [X,Y]\langle\psi,\varphi\rangle\\
&\quad= X(\langle\nabla_Y\psi,\varphi\rangle + \langle\phi,\nabla_Y\varphi\rangle) - Y(\langle\nabla_X\psi,\varphi\rangle + \langle\psi,\nabla_X\varphi\rangle)\\
&\quad= \langle\nabla_X\nabla_Y\psi,\varphi\rangle + \langle\psi,\nabla_X\nabla_Y\varphi\rangle - \langle\nabla_Y\nabla_X\psi,\varphi\rangle - \langle\psi,\nabla_Y\nabla_X\varphi\rangle
\end{aligned}$$

Thus the curvature tensor R of L defined by

$$R(X,Y)\psi = \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi$$

satisfies

$$\langle R(X,Y)\psi,\varphi\rangle + \langle\psi, R(X,Y)\varphi\rangle = 0.$$

In particular this implies that $\langle R(X,Y)\psi,\psi\rangle$ is always imaginary. Thus there is a real-valued 2-form Ω on M such that

$$R(X,Y)\psi = \Omega(X,Y)J\psi.$$

In case L is the tangent bundle TM, the Gaussian curvature K of M at a point p is defined in terms of a unit vector $X \in T_pM$ as

$$K = \langle R(X,JX)JX,X\rangle = -\Omega(X,JX).$$

Thus, if σ denotes the volume form, the curvature 2-form of the tangent bundle is $\Omega = -K\sigma$. We are mostly interested in the case $L = TM^{\otimes n}$, in which case we have $\Omega = -nK\,dA$ and therefore

$$R(X,Y)\psi = -nK\,dA(X,Y)J\psi.$$

The Dirichlet energy of a section ψ of L is defined as

$$E_D(\psi) = \tfrac{1}{2}\int_M |\nabla\psi|^2 dA,$$

where we view $\nabla\psi$ as a 1-form with values in L and the squared norm of such a 1-form ω at a point $p \in M$ is defined as

$$|\omega_p|^2 = |\omega(X)|^2 + |\omega(Y)|^2$$

where $\{X,Y\}$ form an orthonormal basis for T_pM.

Similarly, we define the holomorphic resp. anti-holomorphic energy of ψ as

$$E_H(\psi) = \tfrac{1}{2}\int_M |\bar\partial\psi|^2 dA \qquad E_A(\psi) = \tfrac{1}{2}\int_M |\partial\psi|^2 dA.$$

Theorem: The holomorphic and anti-holomorphic energies of a section ψ are related as

$$E_A(\psi) - E_H(\psi) = \tfrac{1}{2}\int_M nK|\psi|^2 dA - \tfrac{1}{2}\int_{\partial M} \mathrm{Im}\langle\nabla\psi,\psi\rangle.$$

Proof: Because of

$$\mathrm{Re}\langle\nabla\psi,\psi\rangle = \tfrac{1}{2}d\langle\psi,\psi\rangle$$

we can define a real-valued 1-form η on M via

$$\langle\nabla\psi,\psi\rangle = \tfrac{1}{2}d\langle\psi,\psi\rangle + \imath\eta(X).$$

Then for a locally defined unit vector field X we have

$$\begin{aligned}
\imath d\eta(X,JX) &= X\langle\nabla_{JX}\psi,\psi\rangle - (JX)\langle\nabla_X\psi,\psi\rangle - \langle\nabla_{[X,JX]}\psi,\psi\rangle\\
&= \langle R(X,JX)\psi,\psi\rangle + \imath(\langle J\nabla_{JX}\psi,\nabla_X\psi\rangle + \langle\nabla_X\psi, J\nabla_{JX}\psi\rangle)\\
&= \imath(nK|\psi|^2 + |\bar\partial\psi|^2 - |\partial\psi|^2)dA(X,JX).
\end{aligned}$$

Thus

$$d\eta = (nK|\psi|^2 + |\bar\partial\psi|^2 - |\partial\psi|^2)dA$$

and our claim follows by Stokes theorem.

D Integrals

In this section we give a derivation of all the integrals needed in the finite element discretization of the smooth theory, and begin with the construction of the basis sections.

D.1 PL Basis Sections

The PL basis sections Ψ_i are defined by extending the basis X_i from each vertex into the incident triangles through parallel transport along radii, giving us a unit basis section Φ_i supported on all incident triangles. Φ_i is then linearly attenuated with the standard PL hat function at v_i to give us Ψ_i. On a single incident triangle, say t_{ijk}, this amounts to

$$\Psi_i = b_i\Phi_i,$$

where b_i is the standard barycentric coordinate function for v_i in t_{ijk}.

While this procedure, together with the fact that the curvature is defined everywhere, uniquely defines the PL basis sections Ψ_i we do not have a closed form expression for them. Yet we can work out all required integrals in closed form due of our earlier assumption that the curvature is constant in each triangle (Eq. (14)). Specifically, the constancy of curvature over t_{ijk} gives us the holonomy angle $\Omega_{t'}$ (Eq. (13)) along the boundary of any sub-triangle $t' \subset t_{ijk}$ as a fraction of the area:

$$\Omega_{t'} = \int_{t'} nK\,dA = nK_{ijk}|t'|,$$

where $|t'|$ denotes the area of t' (see Eq. (14)).

This brings us to the starting position.

D.2 Mass Matrix (L_2 Metric)

Consider the Hermitian product (complex anti-linear resp. linear in the left resp. right factor) of basis sections restricted to the triangle t_{ijk}

$$\langle\langle \Psi_j, \Psi_k \rangle\rangle_{ijk} := \int_{t_{ijk}} \langle \Psi_j, \Psi_k \rangle dA$$

These determine the L_2 metric on the space of PL sections over a triangle and define the local mass matrix.

Let v be inside t_{ijk} with barycentric coordinates b_i, b_j, b_k

$$v = b_i v_i + b_j v_j + b_k v_k,$$

and consider the integrand of the mass matrix as a function of v

$$\langle \Psi_j(v), \Psi_k(v) \rangle = b_j b_k \langle \Phi_j(v), \Phi_k(v) \rangle. \tag{21}$$

We know that Φ_j is parallel along e_{jk}, implying

$$\Phi_j(v_k) = r_{jk} \Phi_k(v_k)$$

(*cf.* Eq. (2); Fig. 17). Moreover Φ_k is parallel along the ray from

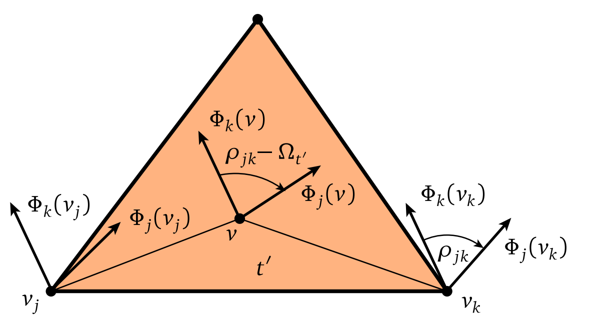

Figure 17: *The angle between two basis sections at some point v in t_{ijk} can be deduced from the holonomy $\Omega_{t'}$ of the sub-triangle $t' = \{v_k, v, v_j\}$ and the known transport along e_{jk}.*

v_k to v and Φ_j along the ray from v to v_j. Since parallel transport around $t' = \{v_k, v, v_j\}$, *i.e.*, from v_k to v on to v_j and finally back to v_k recovers the holonomy angle of t' we get

$$\langle \Phi_j(v), \Phi_k(v) \rangle r_{jk} = e^{\iota \Omega_{t'}} = e^{\iota \Omega_{ijk} b_i}. \tag{22}$$

(using Eqs. (13) and (14). Putting Eqs. (21) and (22) together and integrating symbolically we get

$$\langle\langle \Psi_j, \Psi_k \rangle\rangle_{ijk} = \bar{r}_{jk} |t_{ijk}| \frac{6e^{\iota\Omega_{ijk}} - 6 - 6\iota\Omega_{ijk} + 3\Omega_{ijk}^2 + \iota\Omega_{ijk}^3}{3\Omega_{ijk}^4}.$$

For the product of a basis section with itself the curvature does not play a role and one obtains $\|\Psi_i\|_{ijk}^2 = |t_{ijk}|/6$.

This completely determines the metric on the space of basis n-vector fields.

D.3 Dirichlet Energy

To compute the Dirichlet energy we need the covariant derivatives of Φ_j and Φ_k. To this end we will employ a particular (linear) parameterization $f(x,y)$ of the embedding p_{ijk} of t_{ijk} with $f(0,0) = p_i$, $f(1,0) = p_j$ and $f(0,1) = p_k$, and will treat Φ and Ψ as defined on the image of f in this section. Denote by ∂_x and ∂_y the tangent coordinate frame corresponding to the coordinates x, y (Fig. 18).

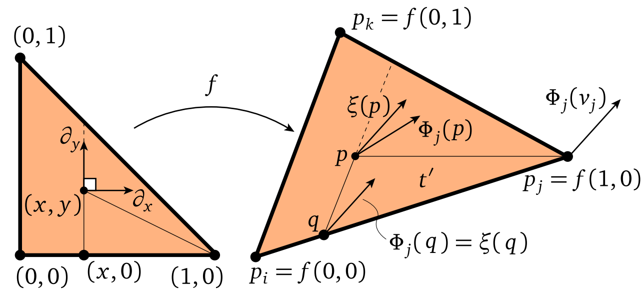

Figure 18: *Using the parameterization f we compute the covariant derivative of a basis n-vector field Φ_j using the local section ξ.*

To compute $\nabla_{\partial_y} \Phi_j$ at some point p in the interior of the triangle, let ξ be a section on the triangle t_{ijk} that agrees with Φ_j for $y = 0$ and is parallel along the lines $\{x = \text{const}\}$, giving $\nabla_{\partial_y} \xi = 0$. For $p = f(x,y)$ let $q := f(x,0)$, then the holonomy around t_{pqj} is $\Omega_{pqj} = (1-x)y\Omega_{ijk}$ and consequently

$$e^{\iota(1-x)y\Omega_{ijk}} \xi(p) = \Phi_j(p).$$

This gives us

$$\nabla_{\partial_y} \Phi_j = (\partial_y e^{\iota(1-x)y\Omega_{ijk}}) \xi = \iota(1-x)\Omega_{ijk} \Phi_j.$$

The derivative $\nabla_{\partial_x} \Phi_k$ follows immediately through interchange of x and y. Taking account of the orientation, we obtain

$$\nabla_{\partial_x} \Phi_k = -\iota(1-y)\Omega_{ijk} \Phi_k.$$

Since by construction Φ_j is parallel along rays from v_j and Φ_k along rays from v_k we get a linear relationship between the covariant derivative of Φ_j with respect to ∂_x and ∂_y (and similarly for Φ_k)

$$0 = (1-x)\nabla_{\partial_x}\Phi_j - y\nabla_{\partial_y}\Phi_j \quad 0 = x\nabla_{\partial_x}\Phi_k + (y-1)\nabla_{\partial_y}\Phi_k$$

which give us all the covariant derivatives for Φ_j and Φ_k. For $\Psi_j = x\Phi_j$ and $\Psi_k = y\Phi_k$ this results in

$$\begin{aligned} \nabla_{\partial_x}\Psi_j &= (1 + \iota\Omega_{ijk}xy)\Phi_j, & \nabla_{\partial_y}\Psi_j &= \iota\Omega x(1-x)\Phi_j, \\ \nabla_{\partial_x}\Psi_k &= -\iota\Omega_{ijk}y(1-y)\Phi_k, & \nabla_{\partial_y}\Psi_k &= (1 - \iota\Omega_{ijk}xy)\Phi_k. \end{aligned}$$

To simplify the computation of the integrals we now switch from the basis $\{\partial_x, \partial_y\}$ to an orthogonal basis $\{E_1, E_2\}$. Letting

$$g_{11} = |p_j - p_i|^2, \quad g_{12} = \langle p_j - p_i, p_k - p_i \rangle = g_{21}, \quad g_{22} = |p_k - p_i|^2,$$

the orthogonal basis follows as

$$E_1 = \frac{1}{\sqrt{g_{11}}}\partial_x, \quad E_2 = \frac{1}{2|t_{ijk}|\sqrt{g_{11}}}(g_{11}\partial_y - g_{12}\partial_x).$$

With respect to this basis we get

$$\nabla_{E_1}\Psi_j = \tfrac{1}{\sqrt{g_{11}}}(1+\imath\Omega_{ijk}xy)\Phi_j,$$
$$\nabla_{E_2}\Psi_j = \tfrac{1}{2|t_{ijk}|\sqrt{g_{11}}}(-g_{12}+\imath\Omega_{ijk}(g_{11}x(1-x)-g_{12}xy))\Phi_j,$$
$$\nabla_{E_1}\Psi_k = \tfrac{1}{\sqrt{g_{11}}}(-\imath\Omega_{ijk}y(1-y))\Phi_k,$$
$$\nabla_{E_2}\Psi_k = \tfrac{1}{2|t_{ijk}|\sqrt{g_{11}}}(g_{11}+\imath\Omega_{ijk}(g_{12}y(1-y)-g_{11}xy))\Phi_k.$$

With this we have all the necessary components to compute the Dirichlet products and integrate them to yield

$$\begin{aligned}\langle\langle\nabla\Psi_j,\nabla\Psi_j\rangle\rangle_{ijk} =& \tfrac{1}{4|t_{ijk}|}\left[g_{22}+\Omega_{ijk}^2\tfrac{3g_{11}-3g_{12}+g_{22}}{90}\right],\\ \langle\langle\nabla\Psi_j,\nabla\Psi_k\rangle\rangle_{ijk} =& \tfrac{\bar r_{jk}}{|t_{ijk}|\Omega_{ijk}^4}[(3g_{11}+4g_{12}+3g_{22})\\ &+\imath\Omega_{ijk}(g_{11}+g_{12}+g_{22})-\imath\Omega_{ijk}^3\tfrac{g_{12}}{6}\\ &+\Omega_{ijk}^4\tfrac{g_{11}-2g_{12}+g_{22}}{24}-\imath\Omega_{ijk}^5\tfrac{g_{11}-2g_{12}+g_{22}}{60}\\ &+(-(3g_{11}+4g_{12}+3g_{22})\\ &+\imath\Omega_{ijk}(2g_{11}+3g_{12}+2g_{22})\\ &+\Omega_{ijk}^2\tfrac{g_{11}+2g_{12}+g_{22}}{2})e^{\imath\Omega_{ijk}}].\end{aligned}$$

D.4 Boundary Terms

The proof in App. C applies to a single triangle t and we only need to compute the boundary term, since the $nK\|\Psi\|_{ijk}^2$ term is the curvature weighted mass matrix (App. D.2).

We begin by observing that the 1-form $\langle\nabla\Psi_j,\Psi_k\rangle$ is nonzero only on e_{jk}. Moreover, since $\langle\nabla\Psi_i,\Psi_i\rangle = d|\Psi_i|^2$, it is real for matching indices giving us

$$\int_{\partial t}\mathrm{Im}\langle\nabla\Psi,\Psi\rangle = \sum_{e_{ij}\in\partial t}\mathrm{Im}\int_{e_{ij}}\left(\bar\alpha_i\alpha_j\langle\nabla\Psi_i,\Psi_j\rangle+\bar\alpha_j\alpha_i\langle\nabla\Psi_j,\Psi_i\rangle\right).$$

Note also that $d\langle\Psi_i,\Psi_j\rangle = \langle\nabla\Psi_i,\Psi_j\rangle+\overline{\langle\nabla\Psi_j,\Psi_i\rangle}$. Since $\Psi_i(v_j)=0=\Psi_j(v_i)$ Stokes' theorem yields

$$0=\int_{\partial e_{ij}}\langle\Psi_i,\Psi_j\rangle = \int_{e_{ij}}\langle\nabla\Psi_i,\Psi_j\rangle+\int_{e_{ij}}\overline{\langle\nabla\Psi_j,\Psi_i\rangle},$$

which we use to simplify the boundary edge sum

$$\int_{\partial t}\mathrm{Im}\langle\nabla\Psi,\Psi\rangle = \sum_{e_{ij}\in\partial t}2\,\mathrm{Im}\left(\bar\alpha_i\alpha_j\int_{e_{ij}}\langle\nabla\Psi_i,\Psi_j\rangle\right).$$

We now turn to an individual edge term. Parameterize e_{jk} with a constant speed $\gamma\colon[0,1]\to M$, *i.e.*, $|\gamma'|=|p_{jk}|$. Due to $db_j(\gamma')=-1$ for b_j the barycentric coordinate function of v_j, and the parallelity of Φ_j along e_{jk}, we get

$$\nabla_{\gamma'}\Psi_j = db_j(\gamma')\Phi_j+b_j\nabla_{\gamma'}\Phi_j = -\Phi_j,$$

and hence

$$\int_{e_{jk}}\langle\nabla\Psi_j,\Psi_k\rangle = \int_0^1\langle\nabla_{\gamma'}\Psi_j,\Psi_k\rangle dt = -\int_0^1 b_k\langle\Phi_j,\Phi_k\rangle dt = -\tfrac{\bar r_{jk}}{2}.$$

This finally yields

$$\int_{\partial t}\mathrm{Im}\langle\nabla\Psi,\Psi\rangle = -\sum_{e_{jk}\in\partial t}\mathrm{Im}\left(\bar r_{jk}\bar\alpha_j\alpha_k\right)$$

Notice that for an edge e_{jk} with two incident triangles (not on the boundary) the corresponding terms from each triangle cancel out, leaving us with only a sum over the boundary of the triangle mesh.

In the flat case ($\Omega_{ijk}=0$) these boundary terms are simply the areas of the range of ψ seen as a complex function, *i.e.*, $2E_H(\psi)=E_D(\psi)-A(\psi(M))$ (compare with [Mullen et al. 2008, Eq. 2], resp. [Pinkall and Polthier 1993, Eq. 2.1]).

D.5 Curvature Alignment

To perform alignment with principal curvature directions we need to find a 2-vector corresponding to these directions (2-vector since principal curvature directions are indistinguishable under rotations by π). In the smooth setting, the principal curvature directions are the eigendirections of the shape operator S, given by the derivative of the Gauss map

$$dN = df\circ S.$$

The *Hopf differential* Q is the trace-free part of the shape operator

$$S = H\cdot I+Q,$$

where $H=(\kappa_1+\kappa_2)/2$ is the mean curvature and I the identity. Q has the principal curvature directions as eigendirections with eigenvalues $(\kappa_1-\kappa_2)/2$ and $(\kappa_2-\kappa_1)/2$. Thus the information provided by Q contains the (unoriented) direction of maximal curvature together with a positive "intensity." Thus Q can be viewed as a 2-vector field.

Using a local complex coordinate z we can write any tangent vector as $a\frac{\partial}{\partial x}$ where x is the real part of z and $a\in\mathbb{C}$. Q is anti-linear as an endomorphism of the tangent space (it anti-commutes with J) and therefore we have

$$Q(a\tfrac{\partial}{\partial x}) = q\bar a\tfrac{\partial}{\partial x}.$$

So with respect to a complex coordinate Q is described by a complex function q.

In the discrete setting the shape operator survives in a distributional sense concentrated along edges [Cohen-Steiner and Morvan 2003]. Consider an edge e and its two incident triangles and map them isometrically to the plane with e along the x-axis, then

$$S_e = \delta_y\beta_e\tfrac{\partial}{\partial y}dy = \delta_y\tfrac{\beta_e}{2}\tfrac{\partial}{\partial x}dz-\delta_y\tfrac{\beta_e}{2}\tfrac{\partial}{\partial x}d\bar z.$$

Here δ_y is the delta distribution along across e, β_e the dihedral angle at e, $dz=dx+\imath dy$, and $d\bar z = dx-\imath dy$. Hence

$$q_e = -\delta_y\tfrac{\beta_e}{2}.$$

Since q exists only as a distribution, we treat it as a functional on the space of smooth sections. In particular we can pair it with each of our PL 2-vector basis sections

$$\tilde q_i := q(\Psi_i) = \sum_{e\ni i}q_e(\Psi_i) = -\tfrac{1}{4}\sum_{e\ni i}r_{ie}\beta_e|p_e|.$$

where the transport coefficient $r_{ie}=e^{\imath 2\theta_i(X_i,e)}$ depends on the rescaled Euclidean angle between X_i and e. Hence the coefficients q of the PL Hopf differential solve the matrix problem

$$Mq=\tilde q.$$